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Space–times homogeneous on a time-like hypersurface

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Abstract. Stationary space–times homogeneous on a time-like hypersurface orthogonal to a space-like congruence are studied. A classification of solutions, including all known solutions, of Einstein's field equations for space–times with vacuum, Einstein space or perfect fluid energy–momentum tensors is given. Time-like hypersurface homogeneous space–times with 'diagonal' metrics are also analysed.

1. Introduction

We shall consider solutions of Einstein's field equations homogeneous on a time-like hypersurface T_3. The homogeneity is manifested by the invariance of geometric objects on the hypersurface under the action of a simply transitive group (or subgroup) of motions G_3. The group of motions is generated by three Killing vectors, one of which must be time-like; therefore the space–time is stationary (or static if the time-like Killing vector is hypersurface orthogonal).

Let $n^a(x)$ be the tangent vector to a space-like congruence, parametrised by x, orthogonal to the family of homogeneous time-like hypersurfaces. Unless otherwise stated we use a tetrad of orthonormal vectors $\{e_a\}$, where $e_1{}^a = n^a$, as our basis vectors. The remaining vectors e_A, $A = 2, 3, 4$, span the tangent space to the orbits of the group of motions. Taking $e_4{}^a e_{4a} = -1$, the orthonormal tetrad components of the metric tensor are $g_{ab} = \mathrm{diag}(1, 1, 1, -1)$.

The conventions stated by Kramer *et al* (1980) are used throughout, taking Einstein's field equations to be $R_{ab} - \frac{1}{2}Rg_{ab} + \Lambda g_{ab} = x_0 T_{ab}$. We limit ourselves to energy–momentum tensors which are of the same form as the tetrad metric components, that is $T_{ab} = \mathrm{diag}(T_{11}, T_{22}, T_{33}, T_{44})$ for diagonal g_{ab}. These include vacuum space–times, perfect fluids (including dust and $T_{ab} = 0$) and Einstein spaces. General electromagnetic and pure radiation fields are not considered.

2. The projection tensor

The tensor $h_{ab} \equiv g_{ab} - n_a n_b$ projects geometric objects onto T_3. One can use h_{ab} to split any tensor field in the space–time into parts parallel and orthogonal to n^a (cf Greenberg 1970). Following techniques used in papers on time-like congruences (e.g. Ellis 1967, Ellis and MacCallum 1973), we can split the covariant derivatives of n_a into symmetric parts by $n_{a;b} = \tilde{\omega}_{ab} + \tilde{\theta}_{ab} + \acute{n}_a n_b$, where $\tilde{\omega}_{ab} = \tilde{\omega}_{[ab]}$, $\tilde{\theta}_{ab} = \tilde{\theta}_{(ab)}$ and $\acute{n}_a = n_{a;b}n^b$, $\tilde{\omega}_{ab}$ and $\tilde{\theta}_{ab}$ being called the vorticity and expansion tensors respectively (the tilde distinguishes these tensors from their counterparts in space-like homogeneous hypersurfaces). Taking the trace and traceless parts of $\tilde{\theta}_{ab}$ gives $\tilde{\theta}_{ab}=\tilde{\sigma}_{ab}+\frac{1}{3}\tilde{\theta}h_{ab}$, where $\tilde{\sigma}^a{}_a=0$ and $\tilde{\theta}=\tilde{\theta}^a{}_a$. We may further define the vorticity vector by $\tilde{\omega}^a\equiv\frac{1}{2}\eta^{abcd}n_b\tilde{\omega}_{cd}$, $\eta_{abcd}\equiv(-g)^{1/2}\varepsilon_{abcd}$, $g=\det(g_{ab})$; then $\tilde{\omega}_{ab}=-\tilde{\omega}^e n^f\eta_{efab}$.

Next we write the Fermi derivatives $\acute{e}_A\cdot e_B$ in terms of the infinitesimal elements, $\tilde{\Omega}^a$, of the Lorentz group in T_3 relating a set of Fermi–Walker propagated axes to the basis tetrad e_a: $\acute{e}_c\cdot e_d\equiv-\eta_{efcd}\tilde{\Omega}^e n^f$, so $\tilde{\Omega}^a=\frac{1}{2}\eta^{abcd}n_b\acute{e}_c\cdot e_d$.

3. The commutators

We define $[e_a, e_b]=\mathscr{L}_{e_b}\equiv D^c{}_{ab}e_c$. This is equivalent to defining $D^c{}_{ab}$ by the two-form equation $d\omega^a\equiv-\frac{1}{2}D^a{}_{bc}\omega^b\wedge\omega^c$, where ω^a are dual to e_a. If $\Gamma^a{}_{bc}$ are the Ricci rotation coefficients, $\Gamma^a{}_{bc}\equiv-\omega^a{}_{i;j}e_b{}^i e_c{}^j=e_b{}^i{}_{;j}\omega^a{}_i e_c{}^j$ ($i, j=1,2,3,4$ refer to a coordinate basis), then it can be shown that $D^c{}_{ab}=\Gamma^c{}_{ba}-\Gamma^c{}_{ab}$. Following previous works, for example Estabrook *et al* (1968) and Ellis and MacCallum (1969), we describe $D^A{}_{BC}$ in terms of a symmetric 'three-tensor' m^{AB} and 'three-vector' b_A by $D^A{}_{BC}\equiv\varepsilon_{BCD}m^{AD}+\delta^A{}_C b_B-\delta^A{}_B b_C$, $\varepsilon_{234}=1=-\varepsilon^{234}$, $A, B, C=2,3,4$. Note that since we are dealing with a time-like hypersurface some equations may have a change of sign compared with analogous equations in the space-like hypersurface case. For example $b_A=\frac{1}{2}D^B{}_{AB}$ but $m^{AB}=-\frac{1}{2}D^{(A}{}_{CD}\varepsilon^{B)CD}$. Thus the commutators can be expressed as

$$[e_1, e_2]=-\acute{n}_2e_1-\tilde{\theta}_{22}e_2-(\tilde{\sigma}_{23}-\tilde{\Omega}^4-\tilde{\omega}^4)e_{33}+(\tilde{\sigma}_{24}+\tilde{\Omega}^3+\tilde{\omega}^3)e_4,$$

$$[e_1, e_3]=-\acute{n}_3e_1-(\tilde{\sigma}_{23}+\tilde{\Omega}^4+\tilde{\omega}^4)e_2-\tilde{\theta}_{33}e_3+(\tilde{\sigma}_{34}-\tilde{\Omega}^2-\tilde{\omega}^2)e_4,$$

$$[e_1, e_4]=-\acute{n}_4e_1-(\tilde{\sigma}_{24}-\tilde{\Omega}^3-\tilde{\omega}^3)e_2-(\tilde{\sigma}_{34}+\tilde{\Omega}^2+\tilde{\omega}^2)e_3+\tilde{\theta}_{44}e_4,$$

$$[e_3, e_4]=2\tilde{\omega}^2e_1+m^{22}e_2+(m^{23}-b_4)e_3+(m^{24}+b_3)e_4,$$

$$[e_4, e_2]=2\tilde{\omega}^3e_1+(m^{23}+b_4)e_2+m^{33}e_3+(m^{34}-b_2)e_4,$$

$$[e_2, e_3]=2\tilde{\omega}^4e_1+(m^{24}-b_3)e_2+(m^{34}+b_2)e_3+m^{44}e_4.$$

Since $n^a(x)$ is hypersurface orthogonal we can write the metric as $ds^2=dx^2+g_{AB}(x)\sigma^A\sigma^B$, σ^A being one-forms in the hypersurface coordinates. In terms of the one-forms ω^a this is just $ds^2=(\omega^1)^2+(\omega^2)^2+(\omega^3)^2-(\omega^4)^2$. Choosing $\omega^1=dx$ gives $'\omega^1=0$ which is equivalent to $D^1{}_{ab}=0$. Thus a consequence of n^a being hypersurface ·thogonal is that $\acute{n}^a=0=\tilde{\omega}^a$.

he reciprocal group

' a set of three basis Killing vectors which generate the simply transitive group of ͡ G_3, then the Lie algebra of the group can be expressed as $[\xi_A, \xi_B]=C^D{}_{AB}\xi_D$ example, Cohn 1957); $C^D{}_{AB}$ are known as the structure constants of the group. ıre vector fields they too can be used to generate a group of transformations, ͻcal group, with group structure 'constants' (see below) $D^A{}_{BC}$. Restricting ıoosing them to be an invariant basis (see Ryan and Shepley 1975), i.e. such $=0$ ($[e_1, \xi_B]=0$ also), and then evaluating the Jacobi identity (ξ_A, e_a, e_b), $^a{}_{bc}$ are constant in each hypersurface but can vary from hypersurface to i.e. $D^a{}_{bc}=D^a{}_{bc}(x)$.

As usual we classify the group types into classes A and B according to whether b^A is zero or non-zero respectively. In both class A and B, for each canonical form of m^{AB} and b_A we can find the associated group types (nine in all) by performing (x-dependent) linear transformations on the basis vectors. Since the group type is independent of any metric on the hypersurface we are not limited to Lorentz transformations (cf beginning of § 6), and we can thereby always reduce m^{AB} and b^A to one of the canonical forms given in the standard tables of (Bianchi) types of three-dimensional real Lie algebras. (It should be noted that since Lie groups are analytic it is not possible to change the group type in a continuous manner; therefore we assume the group type is the same on each hypersurface. This is equivalent to considering only regions of space–time in which there are no discontinuities.) We give as an example the case when $b_A = (b_2, 0, 0)$ and

$$m^{AB} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & m^{34} \\ 0 & m^{34} & m^4 \end{pmatrix}, \qquad b_2 m^{34} m^4 \neq 0$$

(see § 6 (ii)).

The Lie algebra has the following commutators:

$$[e_3, e_4] = 0, \qquad [e_4, e_2] = (m^{34} - b_2)e_4, \qquad [e_2, e_3] = (m^{34} + b_2)e_3 + m^4 e_4.$$

By performing the linear transformations

$$e_2 \to \acute{e}_2 = 1/m^{34} e_2, \qquad e_3 \to \acute{e}_3 = m^{34}/m^4 e_3 + 2e_4, \qquad e_4 \to \acute{e}_4 = m^{34}/m^4 e_3 - e_4,$$

the new basis vectors have the commutators

$$[\acute{e}_3, \acute{e}_4] = 0, \qquad [\acute{e}_4, \acute{e}_2] = -\acute{e}_3 - b_2/m^{34} \acute{e}_4, \qquad [\acute{e}_2, \acute{e}_3] = b_2/m^{34} \acute{e}_3 + \acute{e}_4.$$

Comparing this with the classification tables makes it clear that this is group type VI_h with $h = -(b_2/m^{34})^2$. It can be shown that when $b_A = (b_2, 0, 0)$ and

$$m^{AB} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & m^3 & m^{34} \\ 0 & m^{34} & m^4 \end{pmatrix}$$

that if $m \equiv m^3 m^4 - (m^{34})^2 > 0$ the group type is VII_h, $h = (b_2)^2/m$, and if $m < 0$ the group type is VI_h, $h = (b_2)^2/m$. This is because, as seen in the classification tables, the modulus of the signatu.e of N^{AB} (which is diagonal) differentiates between types VI_h and VII_h. This difference is equivalent to the sign of $N^{22}N^{33}$. Extending this idea to $\begin{pmatrix} m^3 & m^{34} \\ m^{34} & m^4 \end{pmatrix}$ we first need to diagonalise it as $\mathrm{diag}(\lambda_1, \lambda_2)$ and it is easily show that $\lambda_1 \lambda_2 = m^3 m^4 - (m^{34})^2 \equiv m$. So the sign of m discerns between types VI_h and VII_h. The equation $(1-h)D^A{}_{BA}D^E{}_{CE} = -2hD^A{}_{EB}D^E{}_{AC}$, which defines the invariant h, gives $h = (b_2)^2/m$.

5. The Jacobi identities, Ricci tensor and conservation equations

It is useful to note that the Jacobi identities for (e_a, e_b, e_c) are equivalent to the three-form equation $\mathrm{d}^2\omega^a = 0$; the $\omega^b \wedge \omega^c \wedge \omega^d$ component of $\mathrm{d}^2\omega^a$ is just the $\begin{pmatrix} a \\ bcd \end{pmatrix}$ ir standard texts. It is more convenient here to define $J\{a, b\}$ to be equivalent to the Jac identity given (see MacCallum 1979) by the four-form equation $\omega^a \wedge d^2\omega^b = 0$, with the ordering $\omega^1 \wedge \omega^2 \wedge \omega^3 \wedge \omega^4$ taken as positive. Then we can define the symmetric and antisymmetric parts of $J\{a, b\}$ as $J(a, b) \equiv \frac{1}{2}(J\{a, b\} + J\{b, a\})$ and $J[a, b] \equiv \frac{1}{2}(J\{a, b\} - J\{b, a\})$ respectively. The three Jacobi identities $J\{1, A\}$ are in fact given by

$$m^{AB} b_B = 0. \tag{5.1}$$

This is identical to the space-like hypersurface case where $n^{\alpha\beta} a_\beta = 0$ (see Ellis and MacCallum 1969). The other Jacobi identities are tabulated in appendix 1.

The tetrad components of the Riemann tensor, and hence the Ricci tensor, can be calculated from the two-form equation $\Theta^a{}_b \equiv \frac{1}{2} R^a{}_{bcd} \omega^c \wedge \omega^d = d\Gamma^a{}_b + \Gamma^a{}_c \wedge \Gamma^c{}_b$ where $\Gamma^a{}_b = \Gamma^a{}_{bc}\omega^c$. The components of the Ricci tensor are also given in appendix 1.

The contracted Bianchi identities impose on T^{ab} the conservation equations $T^{ab}{}_{;b} = 0$. A perfect fluid with normalised four-velocity u^a has an energy–momentum tensor defined by

$$T^{ab} = (\mu + p) u^a u^b + p g^{ab}, \qquad u^a u_a = -1, \qquad \mu + p \neq 0 \text{ and } \mu > 0,$$

where μ is the density and p the pressure of the fluid. If we take $u^a = e_4{}^a$ then the conservation equations yield

$$p_{,1} - \tilde{\theta}_4(\mu + p) = 0, \tag{5.2}$$

$$(\mu + p)(m^{34} - b_2) = 0, \tag{5.3}$$

$$(\mu + p)(m^{24} + b_3) = 0, \tag{5.4}$$

$$(\mu + p) b_4 = 0. \tag{5.5}$$

Note that (5.4) is trivial since one can perform rotations on the basis vectors which reduce b_3 and m^{24} to zero (see §§ 6 and 8).

One can immediately see from (5.5) that if b^A is time-like or null there can be no perfect fluid solutions. Equation (5.3) gives that there can only be perfect fluid solutions in class B if b^A is space-like with $b_A = (m^{34}, 0, 0)$ (for example Ozsváth's (1965) dust solution [10.29] (see § 9) has $b_2 = m^{34}$). Similarly in class A there are perfect fluid solutions if and only if $m^{34} = 0$. Equation (5.2) gives that Λ-term solutions (i.e. $\mu + p = 0$) must have constant p.

6. Class B, canonical forms of m^{AB} and b^A

In the case of space-like hypersurfaces it is easy to rotate the (space-like) basis vectors so that $n^\alpha{}_\beta$ is diagonal and $a^\beta = (a, 0, 0)$ (see Ellis and MacCallum 1969). This is because the linear transformation $n^\alpha{}_\beta$ is symmetric and will therefore have three eigenvectors, one of which is a^β (from the Jacobi identities), with real eigenvalues. Complications arise when dealing with a time-like hypersurface because b^A can be time-like, space-like or null, and $m^A{}_B$ is not in general symmetric, so cannot always be diagonalised. Any transformations which we apply to b^A and m^{AB} in order to obtain their canonical forms must preserve the metric in the hypersurface $\eta_{AB} = \text{diag}(1, 1, -1)$, i.e. they must be members of the SO(2, 1) Lorentz group.

In class B b^A, and hence m^{AB}, can be reduced to a canonical form as follows.

(i) b^A *time-like.* It is always possible to perform Lorentz transformations on the $\boldsymbol{e}_A$ basis vectors so that $\boldsymbol{e}_4$ is parallel to b^A. The components of b^A are then given by

$b^A = (0, 0, b^4) = (0, 0, -b_4)$, $b_4 \neq 0$. Equation (5.1) gives that $m^{A4} = 0$ so

$$m^{AB} = m^A{}_B = \begin{pmatrix} m^2 & m^{23} & 0 \\ m^{23} & m^3 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

By performing a rotation in the e_2/e_3 plane we are able to reduce m^{AB} to diagonal form, $m^{AB} = \text{diag}(m^2, m^3, m^4)$, without affecting b^A.

(ii) b^A *space-like.* Performing a Lorentz transformation on our basis vectors to make $b^A = (b^2, 0, 0)$, $b^2 = b_2 \neq 0$, we see by (5.1) that

$$m^{AB} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & m^3 & m^{34} \\ 0 & m^{34} & m^4 \end{pmatrix} \Leftrightarrow m^A{}_B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & m^3 & -m^{34} \\ 0 & m^{34} & -m^4 \end{pmatrix}. \tag{6.1}$$

Since $m^A{}_B$ is not symmetric it is in general not possible to diagonalise $m^A{}_B$ and retain real matrix elements. It is however possible, by performing a boost in the e_3/e_4 plane (which does not affect b^A), to reduce m^{AB} to a form in which at least one of m^3, m^4 and m^{34} in (6.1) is zero unless $m^3 = m^4 = \epsilon m^{34}$, $\epsilon = \pm 1$, when it is form invariant under Lorentz transformations.

(iii) b^a *null, real null basis.* When m^{AB} has a (real) null eigenvector, b^A here, any analysis is simplified if we adopt a real null basis as our basis vectors. We choose our new basis $\{\boldsymbol{d}_a\}$ to be related to the orthonormal tetrad by $\boldsymbol{d}_1 = \boldsymbol{e}_1$, $\boldsymbol{d}_2 = \boldsymbol{e}_2$, $\boldsymbol{d}_3 = 2^{-1/2}(\boldsymbol{e}_3 + \boldsymbol{e}_4)$ and $\boldsymbol{d}_4 = 2^{-1/2}(\boldsymbol{e}_4 - \boldsymbol{e}_3)$. The components of the metric tensor with respect to $\{\boldsymbol{d}_a\}$ are

$$g_{ab} \equiv \boldsymbol{d}_a \cdot \boldsymbol{d}_b = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

As before, we limit energy–momentum tensors to ones which have the same form (with respect to the real null basis) as g_{ab}, that is, with respect to $\{d_a\}$,

$$T_{ab} = \begin{pmatrix} T_{11} & 0 & 0 & 0 \\ 0 & T_{22} & 0 & 0 \\ 0 & 0 & 0 & T_{34} \\ 0 & 0 & T_{34} & 0 \end{pmatrix}.$$

This condition is more restrictive than the condition in § 1 but still covers the energy–momentum tensors mentioned in that section apart from perfect fluids ($\mu + p \neq 0$).

The components of geometric objects, with respect to $\{\boldsymbol{d}_a\}$, are worked out in an identical manner as for $\{\boldsymbol{e}_a\}$ above. They are given by the following commutators, in which the $\hat{}$ signifies that the components of those terms are, with respect to $\{\boldsymbol{d}_a\}$,

$$[\boldsymbol{d}_1, \boldsymbol{d}_2] = -\hat{\theta}_2 \boldsymbol{d}_2 + (\hat{\theta}_{24} + \hat{\Omega}^3)\boldsymbol{d}_3 + (\hat{\theta}_{23} - \hat{\Omega}^4)\boldsymbol{d}_4,$$
$$[\boldsymbol{d}_1, \boldsymbol{d}_3] = -(\hat{\theta}_{23} + \hat{\Omega}^4)\boldsymbol{d}_2 + (\hat{\theta}_{34} - \hat{\Omega}^2)\boldsymbol{d}_3 + \hat{\theta}_3 \boldsymbol{d}_4,$$
$$[\boldsymbol{d}_1, \boldsymbol{d}_4] = -(\hat{\theta}_{24} - \hat{\Omega}^3)\boldsymbol{d}_2 + \hat{\theta}_4 \boldsymbol{d}_3 + (\hat{\theta}_{34} + \hat{\Omega}^2)\boldsymbol{d}_4.$$

$[\boldsymbol{d}_A, \boldsymbol{d}_B] = D^C{}_{AB}\boldsymbol{d}_C$ as before with the new m^{AB} and b^A (with respect to $\{\boldsymbol{d}_a\}$) being linearly related to the old m^{AB} and b^A (with respect to $\{e_a\}$). Note that $\hat{\sigma}_{22}=\hat{\theta}_{22}-\frac{1}{3}\hat{\theta}$ and $\hat{\sigma}_{34}=\hat{\theta}_{34}+\frac{1}{3}\hat{\theta}$ but that $\hat{\sigma}_{ab}=\hat{\theta}_{ab}$ for all other a, b. The Ricci tensor and Jacobi identities with respect to $\{d_a\}$ are given in appendix 2.

Once again the Jacobi identities $J\{1, A\}$ give (5.1), $m^{AB}b_B=0$. We are allowed to choose d_3 (say) parallel to the null b^A so $b^A=(0, b^3, 0)\Leftrightarrow b_A=(0, 0, b_4)$ with $b_4=-b^3$. Equation (5.1) then gives $m^{A4}=0$. We have freedom of performing null rotations about d_3 in the hypersurface, that is linear transformations which preserve the inner product of the basis vectors; these are given by $d_2\to \acute{d}_2=d_2+\beta d_3$, $d_3\to\acute{d}_3=d_3$, $d_4\to\acute{d}_4=\beta d_2+b^2/2d_3+d_4$, $\beta\in\mathbb{R}$: then $\acute{g}_{ab}\equiv\acute{d}_a\cdot\acute{d}_b=g_{ab}$. Since b^A is not affected by such transformations we can use them to reduce m^{AB} to one of the forms

$$(\alpha)\ m^{AB}=\mathrm{diag}(m^2, m^3, 0) \qquad \text{or} \qquad (\beta)\ m^{AB}=\begin{pmatrix}0 & m^{23} & 0\\ m^{23} & 0 & 0\\ 0 & 0 & 0\end{pmatrix}. \tag{6.2}$$

7. Analysis of class B

In order to simplify our analysis we wish to use the remaining degrees of freedom in the orientation of the basis vectors to reduce θ_{AB}, $A\neq B$, and Ω^A to zero. This involves analysing the Jacobi identities and field equations. In the interests of brevity we will only give full details of this procedure for the case when b^A is time-like, since analogous procedures are used for space-like and null b^A.

Theorem 7.1. The only space–time for which a time-like b^A is not necessarily a shear eigenvector is that with a group type VI_h, with $h=-\frac{1}{9}$.

Proof. $b_A=(0, 0, b_4)$, so in general

$$m^{AB}=\begin{pmatrix}m^2 & m^{23} & 0\\ m^{23} & m^3 & 0\\ 0 & 0 & 0\end{pmatrix}.$$

The Jacobi identities $J[2, 4]$ and $J[4, 3]$ give $\tilde{\sigma}_{34}=\tilde{\Omega}^2$ and $\tilde{\sigma}_{24}=-\tilde{\Omega}^3$ respectively. Equations (12) and (13) give

$$(12)\quad \tilde{\sigma}_{24}(3b_4+m^{23})+\tilde{\sigma}_{34}m^3=0, \qquad (13)\quad \tilde{\sigma}_{34}(3b_4-m^{23})-\tilde{\sigma}_{24}m^2=0.$$

The determinant of the coefficients of $\tilde{\sigma}_{24}$ and $\tilde{\sigma}_{34}$ in (12) and (13) gives that $9(b_4)^2+m\neq 0$, where here

$$m=\det\begin{pmatrix}m^2 & m^{23}\\ m^{23} & m^3\end{pmatrix}, \Rightarrow\tilde{\sigma}_{24}=0=\tilde{\sigma}_{34} \qquad (\Rightarrow\tilde{\Omega}^2=0=\tilde{\Omega}^3);$$

then b^A is a shear eigenvector. When $9(b_4)^2+m=0$ it is easy to show that the space–time has group type VI_h with $h=(b_4)^2/m=-\frac{1}{9}$.

It can be shown that if $m>0$ the group type is VII_h and if $m<0$ the group type is VI_h (see end of § 4); in both cases $h=(b_4)^2/m$. If $m=0$ but not all m^{AB} are zero the group type is IV and if $m^{AB}=0$ the group type is V. In all cases we can perform x-dependent rotations in the e_2/e_3 plane (this of course leaves m, and hence the group type, unchanged) in order to make $\tilde{\Omega}^4=0$ everywhere and $\tilde{\sigma}_{23}=0$ initially; however (23) gives that in general $\tilde{\sigma}_{23,1}\neq 0$ and so, apart from the specialised cases below, $\tilde{\sigma}_{23}$ cannot be made zero everywhere. When $m^{AB} = \mathrm{diag}(m^2, m^2, 0)$, $m^2 \neq 0$ gives via $J\{2, 2\} - J\{3, 3\}$ that $\tilde{\theta}_2 = \tilde{\theta}_3$, performing an x-dependent rotation makes $\tilde{\Omega}^4 = 0$ everywhere and $\tilde{\sigma}_{23} = 0$ initially, and (23) now ensures that $\tilde{\sigma}_{23}$ is zero throughout space–time. (14) gives $\tilde{\theta}_2 + \tilde{\theta}_3 + 2\tilde{\theta}_4 = 0$. Similarly when $m^{AB} = 0$ an x-dependent rotation reduces $\tilde{\Omega}^4$ and $\tilde{\sigma}_{23}$ to zero everywhere and (14) again gives $\tilde{\theta}_2 + \tilde{\theta}_3 + 2\tilde{\theta}_4 = 0$.

When b^A is space-like we proceed in a similar manner with the exception that, apart from the case when $m^{AB} = 0$, we use x-dependent boosts in the e_3/e_4 plane to try and make $\tilde{\Omega}^2$ and $\tilde{\sigma}_{34}$ zero; these of course preserve the group type.

Theorem 7.2. When b^A is null it is a shear eigenvector.

Proof. Taking $b_A = (0, 0, b_4)$, we can reduce m^{AB} to one of the forms (α) or (β) in (6.2) (this is with respect to $\{\boldsymbol{d}_a\}$). For (α), $J[2, 4]$ implies $\hat{\theta}_3 = 0$ and $J[4, 3]$ implies $\hat{\theta}_{23} = \hat{\Omega}^4$. Substituting in (12) gives $\hat{\theta}_{23} = 0$ ($\Rightarrow \hat{\Omega}^4 = 0$) so b^A is a shear eigenvector. For (β), since the possibility $m^{23} = 0$ (i.e. $m^{AB} = 0$) has been implicitly considered in (α), we may assume $m^{23} \neq 0$. Thus $J\{2, 2\}$ implies $\hat{\theta}_{23} + \hat{\Omega}^4 = 0$. Using this in $J\{4, 3\}$ gives $\hat{\theta}_{23} = 0 = \hat{\Omega}^4$. $J[2, 4]$ gives $\hat{\theta}_3 = 0$. Hence a null b^A is a shear eigenvector.

For a null b^A we are permitted to use x-dependent null rotations and boosts (with respect to the real null basis, boosts are transformations on the basis vectors such that $\boldsymbol{d}_3 \to \acute{\boldsymbol{d}}_3 = A\boldsymbol{d}_3$ and $\boldsymbol{d}_4 \to \boldsymbol{d}_4 = A^{-1}\boldsymbol{d}_4$, $A = A(x)$) in order to make $\hat{\Omega}^A$ zero.

8. Class A, canonical forms of m^{AB}

When $b^A = 0$ the Jacobi identities (5.1) are identically satisfied. Since $m^A{}_B$ is a 3×3 matrix it will have at least one real eigenvalue and eigenvector. We have the freedom of performing Lorentz transformations on our basis vectors in order to bring (the appropriate) one parallel to the eigenvector and so reduce two of m^{AB} to zero. When $m^A{}_B$ has a space-like eigenvector, we Lorentz transform $\boldsymbol{e}_A$ such that $\boldsymbol{e}_2$ is parallel to it; then m^{AB} has the form

$$m^{AB} = \begin{pmatrix} m^2 & 0 & 0 \\ 0 & m^3 & m^{34} \\ 0 & m^{34} & m^4 \end{pmatrix}.$$

As in class B we can perform boosts in the e_3/e_4 plane in order to reduce m^{AB} to a form in which at least one of m^3, m^4 and m^{34} is zero unless $m^3 = m^4 = \epsilon m^{34}$ when it is form invariant under Lorentz transformations. For a time-like eigenvector

$$m^{AB} = \begin{pmatrix} m^2 & m^{23} & 0 \\ m^{23} & m^3 & 0 \\ 0 & 0 & m^4 \end{pmatrix};$$

since $m^A{}_B$ is symmetrical we can rotate in the e_2/e_3 plane in order to make $m^A{}_B = m^{AB}$ diagonal. So when m^{AB} has a space-like or time-like eigenvector it can always be reduced to the general form

$$m^{AB} = \begin{pmatrix} m^2 & 0 & 0 \\ 0 & m^3 & m^{34} \\ 0 & m^{34} & m^4 \end{pmatrix}. \tag{8.1}$$

When the eigenvector is null (taken parallel to $\boldsymbol{d}_3$) then, using a real null basis, the eigenvector equation gives $m^{24}=0=m^4$. If $m^2+m^{34}\neq 0$ we can perform a null rotation about $\boldsymbol{d}_3$ in order to make $m^{23}=0$, leaving $m^{24}=0=m^4$. Transforming to an orthonormal basis, it can be seen that m^{AB} is just in the form (8.1), so this case has already been considered. If $m^2+m^{34}=0$ (assume $m^{23}\neq 0$, otherwise it has been considered above) we can reduce m^3 to zero using a null rotation. So when m^{AB} has a null eigenvector we need only consider

$$m^{AB}=\begin{pmatrix} m^2 & m^{23} & 0 \\ m^{23} & 0 & -m^2 \\ 0 & -m^2 & 0 \end{pmatrix} \qquad \text{(with respect to } \{\boldsymbol{d}_a\}\text{)}.$$

Class A is classified according to whether $M\equiv\det(m^{AB})$ is zero or non-zero. When $M\neq 0$ the group type is VIII or IX and when $M=0$ it is I, II, VI_0 or VII_0. All the details are given in table 1. As in class B, we use any remaining degrees of freedom in order to try and reduce off-diagonal $\tilde{\sigma}_{AB}$ and $\tilde{\Omega}^A$ to zero.

9. The classification tables

Table 1 gives the results of our analysis of class A and table 2 that of class B. In class B we first classify according to whether b^A is time-like, space-like or null (see § 6). In both tables the general m^{AB} and canonical m^{AB} are given for completeness; however analysis was simplified by considering specialisations based on $M\equiv\det(m^{AB})$ or $m\equiv m^2m^3-(m^{23})^2$ (or $m\equiv m^3m^4-(m^{34})^2$ as appropriate) (see end of § 4). Zeros appearing in columns $\tilde{\sigma}_{AB}$ and $\tilde{\Omega}^A$ ($\hat{\theta}_{AB}$ and $\hat{\Omega}^A$ when the real null basis is used) give those components which, by the Jacobi identities and field equations (with our limitation on T_{ab}), must be zero. A '**0**' appearing in these columns means that although the component is not necessarily zero, by the field equations and Jacobi identities we can perform x-dependent rotations, boosts or null rotations which reduce them to zero for all x without affecting any other terms (see § 7). A '—' means that the component cannot (generally) be made zero. In the next column any useful equations, implied by the Jacobi identities or field equations, are tabulated. For each specialisation the appropriate Lie group type is found (see § 4). The final column gives the known solutions (if any), by their reference number given in Kramer *et al* (1980) for each specialisation. The method for classifying known solutions according to our scheme is given in § 10.

10. Known solutions with a G_3 on T_3

We now consider solutions which have a maximal group G_3, or subgroup G_3, acting on a time-like hypersurface. Lists of these solutions appear in Kramer *et al* (1980), although not all of these solutions satisfy our condition on T_{ab}. We refer to these metrics by their equation number in square brackets given in Kramer *et al* (1980) and also write the metrics in their form. The metrics are then written in one of the forms $ds^2=(\omega^1)^2+(\omega^2)^2+(\omega^3)^2-(\omega^4)^2$ or $ds^2=(\omega^1)^2+(\omega^2)^2-2\acute{\omega}^3\acute{\omega}^4$ as appropriate, where ω^a are duals to $\boldsymbol{e}_a$ and $\acute{\omega}^3$, $\acute{\omega}^4$ are dual to $\boldsymbol{d}_3$, $\boldsymbol{d}_4$. Taking $d\omega^a$ gives the $D^a{}_{bc}$ and we classify the metric according to the form of m^{AB} and b_A using our scheme.

Some metrics are easily classified since they are written in standard group type form. For example, Ellis' shear-free dust solution (Ellis 1967), [11.74] which has a G_3II on T_3 is

$$ds^2 = dx^2 + Y^2F^{-2}\,dy^2 + Y^2F^2\,dz^2 - (dt + 2ay\,dz)^2; \qquad [11.74]$$

a is a constant and Y and F are functions of x. Looking at the duals to the reciprocal group generators for type II in standard tables, it can be seen that an obvious choice is

$$\begin{aligned} \omega^1 &= dx & d\omega^1 &= 0 \\ \omega^2 &= YF^{-1}\,dy & d\omega^2 &= (Y_{,x}/Y - F_{,x}/F)\omega^1 \wedge \omega^2 \\ \omega^3 &= YF\,dz \quad \Rightarrow & d\omega^3 &= (Y_{,x}/Y + F_{,x}/F)\omega^1 \wedge \omega^3 \\ \omega^4 &= dt + 2ay\,dz & d\omega^4 &= 2aY^{-2}\omega^2 \wedge \omega^3. \end{aligned}$$

The only non-zero $D^a{}_{bc}$ are $m^4 = -2aY^{-2}$ (b^A and all others m^{AB} are zero, hence type II), $\tilde{\theta}_2 = Y_{,x}/Y - F_{,x}/F$ and $\tilde{\theta}_3 = Y_{,x}/Y + F_{,x}/F$.

Barnes' (1978), type VII_0 solution [11.59] has a slightly disguised form $ds^2 = U^2\,dz^2 + P^2\,dx^2 + A^2[\sin(2\sqrt{2}kx)(du^2 - dv^2) - 2\cos(2\sqrt{2}kx)\,du\,dv]$ [11.59]. U, P and A are functions of z and k is a constant. One chooses

$$\begin{aligned} &\omega^1 = U\,dz, \qquad \omega^2 = P\,dx, \\ &\omega^3 = A[\cos(\sqrt{2}kx - \pi/4)\,du + \sin(\sqrt{2}kx - \pi/4)\,dv], \\ &\omega^4 = A[\sin(\sqrt{2}kx - \pi/4)\,du - \cos(\sqrt{2}kx - \pi/4)\,dv], \end{aligned}$$

giving

$$\begin{aligned} &d\omega^1 = 0, \qquad d\omega^2 = P_{,z}/PU\omega^1 \wedge \omega^2, \\ &d\omega^3 = A_{,z}/AU\omega^1 \wedge \omega^3 - \sqrt{2}k/P\omega^2 \wedge \omega^4, \\ &d\omega^4 = A_{,z}/AU\omega^1 \wedge \omega^4 + \sqrt{2}k/P\omega^2 \wedge \omega^3. \end{aligned}$$

The non-zero $D^a{}_{bc}$ are $m^3 = m^4 = -\sqrt{2}k/P$; hence the solution is type VII_0 since $m^3m^4 > 0$. $\tilde{\theta}_2 = P_{,z}/PU$, $\tilde{\theta}_3 = -\tilde{\theta}_4 = A_{,z}/AU$ ($\tilde{\theta}_3 + \tilde{\theta}_4 = 0$ as required by the Jacobi identities and field equations).

There are some solutions for which the one-forms are most easily spotted by performing a coordinate change in the metric. An example is Ozsváth's (1965) solution [10.30] which has a group type IV acting on T_3:

$$ds^2 = a^2\{-[(b^2-1)/2b]^2 e^{-z}\,dt^2 + (e^{-Fz}\,dy)^2 + dz^2 + (z\,dt - dx)^2 e^{-z}\};$$

a, b and F are constants.

We use its three Killing vectors ∂_t, ∂_x and $\frac{1}{2}t\partial_t + (t + \frac{1}{2}x)\partial_x + Fy\partial_y + \partial_z$ in order to determine the change in coordinates $w = w(y, z)$ and $r = r(y, z)$ which gives the metric in a standard IV form.

The transformations are in fact

$$w = (a/F)\sinh^{-1}(yF\,e^{-Fz}), \qquad r = \tfrac{1}{2}F\ln(e^{2Fz} + F^2y^2).$$

The metric becomes

$$\begin{aligned} ds^2 = dw^2 &+ a^2\cosh^2(aw/F)\,dr^2 + a^2\,e^{-r}[\cosh(aw/F)^{1/F}] \\ &\times\{[r + \ln(\cosh(aw/F))^{-1/F}]\,dt - dx\}^2 \\ &- a^2[(b^2-1)/2b]\,e^{-r}\cosh(aw/F)^{1/F}\,dt^2. \end{aligned}$$

Table 1. Class A (see text for explanation).

m^{AB}	Specialisations	$\tilde\sigma_{23}$	$\tilde\sigma_{24}$	$\tilde\sigma_{34}$	$\tilde\Omega^4$	$\tilde\Omega^3$	$\tilde\Omega^2$	Implied equations	Group type	Known solutions
General:	*1. $M\neq 0$*									
	(a) $m^{34}=0$									
$\begin{pmatrix} m^2 & 0 & 0 \\ 0 & m^3 & m^{34} \\ 0 & m^{34} & m^4 \end{pmatrix}$	$m^2\neq m^3\neq -m^4\neq m^2$	0	0	0	0	0	0		$\begin{cases}\text{IX if } m^{\wedge}{}_A\leq 0\\ \text{VIII otherwise}\end{cases}$	[10.28], [10.27]
	$m^2=m^3\neq -m^4$	**0**	0	0	**0**	0	0	$\tilde\theta_2=\tilde\theta_3$	ditto	[11.43] (IX) [11.4] $k=-1$ (VIII)
$M\equiv\det(m^{AB})$	$m^2\neq m^3=-m^4$	0	0	—	0	0	—		VIII	[11.17]
$m\equiv m^3m^4-(m^{34})^2$	$m^2=m^3=-m^4$	0	0	0	**0**	**0**	**0**	$\tilde\theta_2=\tilde\theta_3=-\tilde\theta_4$	VIII	
	(b) $m^3m^4=0$									
Canonical:	$m^3=0\begin{cases} m^2\neq -m^4\neq 0 \end{cases}$	—	—	$-\hat\Omega^2$	—	—	$-\tilde\sigma_{34}$		VIII	
	$m^3=0\begin{cases} m^2=-m^4 \end{cases}$	0	0	$-\tilde\Omega^2$	0	0	$-\tilde\sigma_{34}$		VIII	
(α) diag(m^2, m^3, m^3)	$m^4=0\begin{cases} m^2\neq m^3\neq 0 \end{cases}$	—	—	$\tilde\Omega^2$	—	—	$\tilde\sigma_{34}$		VIII	
	$m^4=0\begin{cases} m^2=m^3 \end{cases}$	0	0	$\tilde\Omega^2$	0	0	$\tilde\sigma_{34}$		VIII	
(β) general with $m^3m^4=0$	$m^3=0=m^4$								VIII	No solution exists
	(c) $m^3m^4m^{34}\neq 0$	—	—	—	—	—	—		VIII or IX	
	2. $M=0$									
(γ) general with	(a) $m^2=0$									
$m^3=m^4=\epsilon m^{34}$, $\epsilon=\pm 1$	$m>0$	0	0	—	0	0	**0**		VII_0	
	$(m^3=m^4\neq 0, m^{34}=0)$	0	0	0	0	0	0	$\tilde\theta_3+\tilde\theta_4=0$	VII_0	[11.59]
	$m<0$	0	0	—	0	0	**0**		VI_0	
	$(m^3=-m^4\neq 0, m^{34}=0)$	0	0	**0**	0	0	**0**	$\tilde\theta_3+\tilde\theta_4=0$	VI_0	[11.16] $k=0$
	$m=0, m^{AB}\neq 0$	—	—	—			—		II	[11.74]

	$m^{AB}=0$	—	—	—	**0**	**0**	**0**		I	[11.3] $k=0$, [11.16] $k=0$ [11.5], [10.14], [20.7] [10.29], [10.30], [10.31]
	(b) $m^2\neq 0\ (m=0)$ (i) $m^3m^4m^{34}=0$									
	$m^3=m^4=m^{34}=0$	0	0	—	0	0	**0**		II	[11.17]
	$m^4=0=m^{34}$, $m^2\neq m^3$	0	0	0	0	0	0		$m^2m^3<0$ VI$_0$ $m^2m^3<0$ VII$_0$	
	$m^4=0=m^{34}$, $m^2=m^3$	0	0	0	**0**	0	0	$\tilde{\theta}_2=\tilde{\theta}_3$	VII$_0$	[11.3] $k=0$
	$m^3=0=m^{34}$, $m^2+m^4\neq 0$	0	0	0	0	0	0		$m^2m^4<0$ VI$_0$ $m^2m^4>0$ VII$_0$	
	$m^3=0=m^{34}$, $m^2+m^4=0$	0	0	0	0	**0**	0	$\tilde{\theta}_2+\tilde{\theta}_4=0$	VI$_0$	
	(ii) $m^3m^4m^{34}\neq 0$									
	$m^3=m^4=\epsilon m^{34}$	0	0	—	0	0	**0**	$\tilde{\theta}_3+\tilde{\theta}_4+2\epsilon\tilde{\sigma}_{34}=0$	$m^2m^3>0$ VII$_0$ $m^2m^3<0$ VI$_0$	
	(any other forms can always be reduced to case (i))									
		$\hat{\theta}_{23}$	$\hat{\theta}_{24}$	$\hat{\theta}_{34}$	$\hat{\Omega}^4$	$\hat{\Omega}^3$	$\hat{\Omega}^2$			
Real null basis General: $\begin{pmatrix} m^2 & m^{23} & 0 \\ m^{23} & m^3 & m^{34} \\ 0 & m^{34} & 0 \end{pmatrix}$ $m\equiv -(m^{34})^2$	$m^{23}\neq 0,\ m^3=0,$ $m^{34}=-m^2$									
	(i) $m=0\ (m^2=0)$	0	$-\hat{\Omega}^3$	$-\hat{\theta}_2$	0	$-\hat{\theta}_{24}$	**0**	$\hat{\theta}_3=0,\ \hat{\theta}_2+\hat{\theta}_{34}=0$	VI$_0$	
	(ii) $m\neq 0\ (m^2\neq 0)$ (All other cases are considered wrt the orthonormal basis in the above section)	0	—	$-\hat{\theta}_2$	0	—	**0**	$\hat{\theta}_3=0,\ \hat{\theta}_2+\hat{\theta}_{34}=0$	VIII	

Table 2. Class B (see text for explanation).

b^A	m^{AB}	Specialisations	$\tilde{\sigma}_{23}$	$\tilde{\sigma}_{24}$	$\tilde{\sigma}_{34}$	$\tilde{\Omega}^4$	$\tilde{\Omega}^3$	$\tilde{\Omega}^2$	Implied equations	Group type	Known solutions
Null, $b_A=(0,0,b_4)$ Real null basis	General:	$m>0$	—	0	0	**0**	0	0		VII_h, $h=(b_4)^2/m$	
	$\begin{pmatrix} m^2 & m^{23} & 0 \\ m^{23} & m^3 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$(m^2=m^3,\ m^{23}=0)$	0	0	0	**0**	0	0	$\tilde{\theta}_2=\tilde{\theta}_3=-\tilde{\theta}_4$	VII_h, $h=(b_4)^2/m$	[11.5]
		$m<0,\ 9(b_4)^2+m\neq 0$	—	0	0	**0**	0	0		VI_h, $h=(b_4)^2/m$	[11.16] $k=1$ (III)
	$m\equiv m^2m^3-(m^{23})^2$	$m<0,\ 9(b_4)^2+m=0$	—	—	—	**0**	—	—		VI_h, $h=-\frac{1}{9}$	
	Canonical:	$m=0,\ m^{AB}\neq 0$	—	0	0	**0**	0	0		IV	
	$\mathrm{diag}(m^2, m^3, 0)$	$m^{AB}=0$	**0**	0	0	**0**	0	0	$\tilde{\theta}_2+\tilde{\theta}_3+2\tilde{\theta}_4=0$	V	[11.5]
	General:										
	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & m^3 & m^{34} \\ 0 & m^{34} & m^4 \end{pmatrix}$	$m>0$	0	0	—	0	0	**0**		VII_h, $h=(b_2)^2/m$	[10.14], [10.31]
		$m<0,\ 9(b_2)^2+m\neq 0$	0	0	—	0	0	**0**		VI_h, $h=(b_2)^2/m$	{[11.3] $k=-1$ (III); [11.16] $k=-1, 1$ (III); [11.17] (III), [11.21] (III); [10.29], [11.17]
Space-like $b_A=(b_2, 0\ 0)$	$m\equiv m^3m^4-(m^{34})^2$	$m<0,\ 9(b_2)^2+m=0$	—	—	—	—	—	**0**		VI_h, $h=-\frac{1}{9}$	
	Canonical:										
	(α) $\mathrm{diag}(0, m^3, m^4)$	$m=0,\ m^{AB}\neq 0$	0	0	—	0	0	**0**		IV	[10.30]
	(β) general with $m^3m^4=0$	$m^{AB}=0$	0	0	**0**	0	0	**0**	$2\tilde{\theta}_2-\tilde{\theta}_3+\tilde{\theta}=0$	V	[11.17]
	(γ) general with $m^3=m^4=\epsilon m^{34}$										
			$\hat{\theta}_{23}$	$\hat{\theta}_{24}$	$\hat{\theta}_{34}$	$\hat{\Omega}^4$	$\hat{\Omega}^3$	$\hat{\Omega}^2$			
Time-like $b_A=(0,0,b_4)$	General:										
	$\begin{pmatrix} m^2 & m^{23} & 0 \\ m^{23} & m^3 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$m>0$	0	$\hat{\Omega}^3$	—	0	$-\hat{\theta}_{24}$	—	$\hat{\theta}_3=0$	VII_h, $h=(b_4)^2/m$	
		$m<0$	0	$-\hat{\Omega}^3$	—	0	$-\hat{\theta}_{24}$	—	$\hat{\theta}_3=0$	VI_h, $h=(b_4)^2/m$	
		$m=0,\ m^{AB}\neq 0$									
	$m\equiv m^2m^3-(m^{23})^2$	(i) $\mathrm{diag}(m^2, 0, 0)$	0	$-\hat{\Omega}^3$	—	0	$-\hat{\theta}_{24}$	**0**	$\hat{\theta}_3=0$	IV	
	Canonical:										
	(α) $\mathrm{diag}(m^2, m^3, 0)$	(ii) $\mathrm{diag}(0, m^3, 0)$	0	—	$-\hat{\theta}_2$	0	—	**0**	$\hat{\theta}_3=0,\ \hat{\theta}_2+\hat{\theta}_{34}=0$	IV	
	(β) general with $m^2=0=m^3$	$m^{AB}\neq 0$	0	—	$\hat{\theta}_2$	0	**0**	**0**	$\hat{\theta}_3=0,\ \hat{\theta}_2+\hat{\theta}_{34}=0$	V	

Then we choose

$$\omega^1 = \mathrm{d}w, \qquad \omega^2 = a\cosh(aw/F)\,\mathrm{d}r,$$
$$\omega^3 = a\{[r + \ln(\cosh(aw/F)^{-1/F})]\,\mathrm{d}t - \mathrm{d}x\}\,\mathrm{e}^{-r/2}\cosh(aw/F)^{F/2},$$
$$\omega^4 = a[(b^2-1)/2b]^{1/2}\,\mathrm{e}^{-r/2}\cosh(aw/F)^{F/2}\,\mathrm{d}t.$$

$\mathrm{d}\omega^a$ give that the only non-zero $D^a{}_{bc}$ are b_2 (so b^A is space-like), m^3 (type IV), $\tilde\theta_3 = -\tilde\theta_4$, $\tilde\theta_2$ and $\tilde\sigma_{34} = \tilde\Omega^2$. All the known solutions are classified in tables 1 and 2.

11. 'Diagonal' metrics

We consider metrics of the form $\mathrm{d}s^2 = \mathrm{d}x^2 + A^2(x)\sigma^2\sigma^2 + B^2(x)\sigma^3\sigma^3 - C^2(x)\sigma^4\sigma^4$, σ^A being one-forms in the time-like hypersurface coordinates such that $\mathrm{d}\sigma^A = -\frac{1}{2}C^A{}_{BC}\sigma^B \wedge \sigma^C$, with the Ricci tensor R_{ab} also being diagonal (see MacCallum 1972). It can easily be seen that the duals, ω^a, to the orthonormal tetrad are just $\omega^a = (\mathrm{d}x, A\sigma^2, B\sigma^3, C\sigma^4)$. The two-form $\mathrm{d}\omega^a$ then has $\tilde\Omega^a = 0$ and $\tilde\sigma_{ab} = 0$, $a \neq b$.

Theorem 10.1. In space–times with 'diagonal' metrics b^A, in class B, is always a shear eigenvector.

Proof. By contradiction (see MacCallum 1972). $\tilde\theta^A{}_B = \mathrm{diag}(\tilde\theta_2, \tilde\theta_3, -\tilde\theta_4)$; assume that $\tilde\theta^A{}_B$ are not all equal, otherwise any vector would be a shear eigenvector. Consider any pair of $\tilde\theta^A{}_B$ being equal.

(i) By (14), $\tilde\theta_2 = \tilde\theta_3 \Rightarrow \tilde\theta_2 = -\tilde\theta_4$ or $b_4 = 0$.

(ii) By (13), $\tilde\theta_2 = -\tilde\theta_4 \Rightarrow \tilde\theta_3 = -\tilde\theta_4$ or $b_3 = 0$.

(iii) By (12), $\tilde\theta_3 = -\tilde\theta_4 \Rightarrow \tilde\theta_2 = -\tilde\theta_4$ or $b_2 = 0$.

In all cases b^A is a shear eigenvector. We conclude that if b^A is not a shear eigenvector $\tilde\theta_2$, $\tilde\theta_3$ or $-\tilde\theta_4$ cannot be equal. If only one component of b^A is non-zero, b^A is obviously a shear eigenvector, so we must consider cases when at least two components of b^A are non-zero. First note the following equations:

$$(23)_{,1} + (\tilde\theta_2 + \tilde\theta_3)(23) + 2m^2(14) \Rightarrow (b_4 - m^{23})(m^2 + m^3)(\tilde\theta_3 + \tilde\theta_4) = 0,$$
$$(23)_{,1} + (\tilde\theta_2 + \tilde\theta_3)(23) - 2m^3(14) \Rightarrow (b_4 + m^{23})(m^2 + m^3)(\tilde\theta_2 + \tilde\theta_4) = 0, \qquad (10.1a)$$

$$(24)_{,1} + (\tilde\theta_2 + \tilde\theta_4)(24) + 2m^2(13) \Rightarrow (b_3 + m^{24})(m^2 - m^4)(\tilde\theta_3 + \tilde\theta_4) = 0,$$
$$(24)_{,1} + (\tilde\theta_2 + \tilde\theta_4)(24) + 2m^4(13) \Rightarrow (b_3 - m^{24})(m^2 - m^4)(\tilde\theta_2 - \tilde\theta_3) = 0, \qquad (10.1b)$$

$$(34)_{,1} + (\tilde\theta_3 + \tilde\theta_4)(34) - 2m^3(12) \Rightarrow (b_2 - m^{34})(m^3 - m^4)(\tilde\theta_2 + \tilde\theta_4) = 0,$$
$$(34)_{,1} + (\tilde\theta_3 + \tilde\theta_4)(34) - 2m^4(12) \Rightarrow (b_2 + m^{34})(m^3 - m^4)(\tilde\theta_2 - \tilde\theta_3) = 0. \qquad (10.1c)$$

If, say, $b_3b_4 \neq 0$ then (10.1a) gives $m^2 + m^3 = 0$ and (10.1b) gives $m^2 = m^4$. Equation (10.1c) implies either (i) $b_2 = 0 = m^{34}$ or (ii) $m^3 = m^4$, so $m^2 = m^3 = m^4 = 0$. For (i), (5.1) implies $m^3 = 0 = m^4$; then (10.1a) or (10.1b) gives $m^2 = 0$. With $A = 2$ in (5.1) we have

$$m^{23}b_3 + m^{24}b_4 = 0. \qquad (10.2)$$

Subtracting (34) gives $m^{23}(b_3 - m^{24}) = 0$. If $b_3 = m^{24}$ then (13) implies $\tilde\theta_3 + \tilde\theta_4 = 0$, a contradiction. If $m^{23} = 0$, (10.2) implies $m^{24} = 0$ so $m^{AB} = 0$, but subtracting (13) from (14) also gives $\tilde\theta_3 + \tilde\theta_4 = 0$. For (ii), adding (5.1) to the field equations (AB), $A \neq B$, once again leads to contradictions. Other such contradictions arise by considering $b_2b_4 \neq 0$ and $b_2b_3 \neq 0$. So b^A is a shear eigenvector.

When b^A is space-like theorem 10.1 ensures that it is possible to transform $b_A \to b'_A = (b'_2, 0, 0)$ without affecting the diagonal form of $\tilde{\theta}_{AB}$. (5.1) gives $m^{2A} = 0$ and using these in (34) gives

$$b_2(m^3 + m^4) + m^{34}(m^3 - m^4) = 0. \tag{10.3}$$

Differentiating (10.3) and substituting for the derivatives of m^{AB} and b_A from the Jacobi identities, then adding $2\tilde{\theta}_2$ (34), gives

$$(\tilde{\theta}_3 + \tilde{\theta}_4)[(m^4 - m^3)b_2 - (m^3 + m^4)m^{34}] = 0. \tag{10.4}$$

If $(m^4 - m^3)b_2 = (m^3 + m^4)m^{34}$, multiplying by $(m^4 - m^3)$, noting that, by (10.3), $m^3 = m^4$ implies $m^3 = m^4 = 0$, gives $b_2(m^4 - m^3)^2 = b_2(m^3 + m^4)^2$. Thus (10.4) gives either (i) $(m^4 - m^3)^2 = (m^3 + m^4)^2 \Leftrightarrow m^3m^4 = 0$, or (ii) $\tilde{\theta}_3 + \tilde{\theta}_4 = 0$ which, by (12), implies $\tilde{\theta}_2 = \tilde{\theta}_3 = -\tilde{\theta}_4$.

In case (i) if $m^3 = 0$, (10.3) gives either $b_2 = m^{34}$ or $m^4 = 0$. If $b_2 = m^{34}$, (12) implies $\tilde{\theta}_2 = \tilde{\theta}_3$, substituting in R_{22} and R_{33} gives $R_{22} = R_{33}$ and so the space–time is locally rotationally symmetric (LRS) (see Stewart and Ellis 1968, Ellis and MacCallum 1969). In case (i) when $m^4 = 0$, (10.3) implies $b_2 = -m^{34}$ or $m^3 = 0$. If $b_2 = -m^{34}$, (12) implies $\tilde{\theta}_2 = -\tilde{\theta}_4$ and then $R_{22} = -R_{44}$ and so the space–time is locally boost symmetric (LBS) (see MacCallum 1980), i.e. the isotropy is a boost in a time-like two-surface, here the e_2/e_4 plane. We conclude that in case (i) when $m^{34} \neq 0$ either

$$m^{AB} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & m^{34} \\ 0 & m^{34} & 0 \end{pmatrix}$$

and the group type is VI_h with $h = -(b_2)^2/(m^{34})^2$ or the space–time is LRS or LBS with group type III. If $m^{34} = 0$ the group type is III or V depending on the values of m^3 and m^4.

Similarly we can show that if b^A is time-like, $b_A \to b'_A = (0, 0, b_4)$, by (5.1) this implies $m^{A4} = 0$, with $\tilde{\theta}_{AB}$ remaining diagonal; the Jacobi identity and (23) give

$$b_4(m^2 - m^3) + m^{23}(m^2 + m^3) = 0. \tag{10.5}$$

As in the space-like case this leads to either (i) $(m^2 + m^3)^2 = (m^2 - m^3)^2 \Leftrightarrow m^2m^3 = 0$ or (ii) $\tilde{\theta}_2 = \tilde{\theta}_3$, (14) implies $\tilde{\theta}_2 = \tilde{\theta}_3 = -\tilde{\theta}_4$.

In case (i) if $m^2 = 0$, (10.5) gives either $b_4 = m^{23}$ or $m^3 = 0$. If $b_4 = m^{23}$, (14) implies $\tilde{\theta}_2 = -\tilde{\theta}_4$ and it can be seen that $R_{22} = -R_{44}$ and so we have LBS. If in case (i) $m^3 = 0$, (10.5) gives either $b_4 = -m^{23}$ or $m^2 = 0$. By (14), $b_4 = -m^{23}$ implies $\tilde{\theta}_3 = -\tilde{\theta}_4$, and then $R_{33} = -R_{44}$ and so the space is LBS. In case (i) we conclude that if $m^{23} \neq 0$ either

$$m^{AB} = \begin{pmatrix} 0 & m^{23} & 0 \\ m^{23} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

with group type VI_h, $h = -(b_4)^2/(m^{23})^2$ or the space–time is LBS with group type III. If $m^{23} = 0$ the group type is III or V according to m^2 and m^3.

When b^A is null it is quicker to use an orthogonal basis so $b^A = (0, -b, b) = bd_3$, $b \neq 0$. Theorem 10.1, which once again ensures we can write b^A in this fashion without affecting the diagonality of $\tilde{\theta}_{ab}$, gives $\tilde{\theta}_3 + \tilde{\theta}_4 = 0$. Equation (5.1) implies $m^{23} = -m^{24}$

and $m^3 = -m^{34} = m^4$. Using these in (13) or (14) gives $(\tilde{\theta}_2 - \tilde{\theta}_3)(b - m^{23}) = 0$, so either (i) $b = m^{23}$ or (ii) $\tilde{\theta}_2 = \tilde{\theta}_3 = -\tilde{\theta}_4$. If $b = m^{23}$ then (34) gives $m^2 m^3 = 0$. It can then be seen that $R_{33} = -R_{44}$ and so the space–time is LBS with group type IV or V.

For b^A space-like, time-like and null the metric for cases (ii) is just $ds^2 = dx^2 + A^2(x)(\sigma^2\sigma^2 + \sigma^3\sigma^3 - \sigma^4\sigma^4)$, where σ^A are one-forms in the hypersurface coordinates.

12. Conclusions

Space–times homogeneous on a time-like hypersurface orthogonal to a space-like congruence were studied using an orthonormal and a real null tetrad technique in analogy to the study of space-like hypersurfaces.

The space–time models fall into two classes, class A and class B, depending on whether b^A (part of the reciprocal group structure 'constants' as defined in § 3) is zero or non-zero respectively. In class B a time-like b^A is necessarily a shear eigenvector unless the Bianchi type is VI_h, $h = -\frac{1}{9}$. When b^A is null it is always a shear eigenvector.

The canonical forms of m^{AB} (defined in § 3) are given. In class A the models are broadly classified into two classes defined by M, where $M \equiv \det(m^{AB})$, being non-zero or zero. Class B models are classified according to whether b^A is time-like, space-like or null and then according to the sign of m, where $m = m^2 m^3 - (m^{23})^2$ or $m = m^3 m^4 - (m^{34})^2$ as appropriate. The classification tables included all known solutions, and the method of classifying known solutions using our scheme was given in § 10.

'Diagonal' metrics are investigated. For these metrics b^A is always a shear eigenvector. Finally, for class B the 'diagonal' solutions fall into two categories. Either they are of types III, IV, V or VI_h (the first includes those solutions which have a two-surface which is locally rotationally symmetric or locally boost symmetric) or they have $\tilde{\theta}_2 = \tilde{\theta}_3 = -\tilde{\theta}_4$ so the metric can be written as $ds^2 = dx^2 + A^2(x)(\sigma^2\sigma^2 + \sigma^3\sigma^3 - \sigma^4\sigma^4)$, where σ^A are one-forms in the hypersurface coordinates.

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Appendix 1. Jacobi identities for orthonormal tetrad basis

We define $J\{a, b\}$ to be equivalent to the Jacobi identity given by the equation $\omega^a \wedge d^2\omega^b = 0$ with the ordering $\omega^1 \wedge \omega^2 \wedge \omega^3 \wedge \omega^4$ taken as positive. The symmetrised $J(a, b)$ and antisymmetric $J[a, b]$ are defined as usual. The non-trivial Jacobi identities are:

$$J\{1, A\} \quad m^{AB} b_B = 0 \qquad A, B = 2, 3, 4,$$

$$J\{2, 2\} \quad m^2{}_{,1} + m^2(\tilde{\theta}_3 - \tilde{\theta}_2 - \tilde{\theta}_4) - 2m^{23}(\tilde{\sigma}_{23} + \tilde{\Omega}^4) - 2m^{24}(\tilde{\sigma}_{24} - \tilde{\Omega}^3) = 0,$$

$$J\{3, 3\} \quad m^3{}_{,1} + m^3(\tilde{\theta}_2 - \tilde{\theta}_3 - \tilde{\theta}_4) - 2m^{23}(\tilde{\sigma}_{23} - \tilde{\Omega}^4) - 2m^{34}(\tilde{\sigma}_{34} + \tilde{\Omega}^2) = 0,$$

$J\{4,4\}$ $m^{4}{}_{,1}+m^{4}(\tilde{\theta}_2+\tilde{\theta}_3+\tilde{\theta}_4)+2m^{24}(\tilde{\sigma}_{24}+\tilde{\Omega}^3)+2m^{34}(\tilde{\sigma}_{34}-\tilde{\Omega}^2)=0,$

$J(2,3)$ $m^{23}{}_{,1}-m^{23}\tilde{\theta}_4-m^{24}(\tilde{\sigma}_{34}+\tilde{\Omega}^2)$
$-m^{34}(\tilde{\sigma}_{24}-\tilde{\Omega}^3)-m^{2}(\tilde{\sigma}_{23}-\tilde{\Omega}^4)-m^{3}(\tilde{\sigma}_{23}+\tilde{\Omega}^4)=0,$

$J(2,4)$ $m^{24}{}_{,1}+m^{24}\tilde{\theta}_3+m^{23}(\tilde{\sigma}_{34}-\tilde{\Omega}^2)$
$-m^{34}(\tilde{\sigma}_{23}+\tilde{\Omega}^4)+m^{2}(\tilde{\sigma}_{24}+\tilde{\Omega}^3)-m^{4}(\tilde{\sigma}_{24}-\tilde{\Omega}^3)=0,$

$J(3,4)$ $m^{34}{}_{,1}+m^{34}\tilde{\theta}_2-m^{24}(\tilde{\sigma}_{23}-\tilde{\Omega}^4)+m^{23}(\tilde{\sigma}_{24}+\tilde{\Omega}^3)$
$+m^{3}(\tilde{\sigma}_{34}-\tilde{\Omega}^2)-m^{4}(\tilde{\sigma}_{34}+\tilde{\Omega}^2)=0,$

$J[3,2]$ $b_{4,1}+b_3(\tilde{\sigma}_{34}+\tilde{\Omega}^2)+b_2(\tilde{\sigma}_{24}-\tilde{\Omega}^3)-\tilde{\theta}_4 b_4=0,$

$J[2,4]$ $b_{3,1}-b_4(\tilde{\sigma}_{34}-\tilde{\Omega}^2)+b_2(\tilde{\sigma}_{23}+\tilde{\Omega}^4)+\tilde{\theta}_3 b_3=0,$

$J[4,3]$ $b_{2,1}-b_4(\tilde{\sigma}_{24}+\tilde{\Omega}^3)+b_3(\tilde{\sigma}_{23}-\tilde{\Omega}^4)+\tilde{\theta}_2 b_2=0.$

Ricci tensor (wrt orthonormal basis)

$R_{11}=-\tilde{\theta}_{,1}-(\tilde{\theta}_2)^2-(\tilde{\theta}_3)^2-(\tilde{\theta}_4)^2+2[(\tilde{\sigma}_{34})^2+(\tilde{\sigma}_{24})^2-(\tilde{\sigma}_{23})^2],$

$R_{22}=-\tilde{\theta}_{2,1}-\tilde{\theta}_2\tilde{\theta}+2\tilde{\sigma}_{23}\tilde{\Omega}^4+2\tilde{\sigma}_{24}\tilde{\Omega}^3+2b_3m^{24}$
$+2b_4m^{23}-2(b_2)^2-2(b_3)^2+2(b_4)^2-2(m^{34})^2+\tfrac{1}{2}[(m^3+m^4)^2-(m^2)^2],$

$R_{33}=-\tilde{\theta}_{3,1}-\tilde{\theta}_3\tilde{\theta}-2\tilde{\sigma}_{23}\tilde{\Omega}^4-2\tilde{\sigma}_{34}\tilde{\Omega}^2-2b_2m^{34}$
$-2b_4m^{23}-2(b_2)^2-2(b_3)^2+2(b_4)^2-2(m^{24})^2+\tfrac{1}{2}[(m^2+m^4)^2-(m^3)^2],$

$R_{44}=-\tilde{\theta}_{4,1}-\tilde{\theta}_4\tilde{\theta}-2\tilde{\sigma}_{34}\tilde{\Omega}^2+2\tilde{\sigma}_{24}\Omega^3-2b_2m^{34}$
$+2b_3m^{24}+2(b_2)^2+2(b_3)^2-2(b_4)^2-2(m^{23})^2+\tfrac{1}{2}[(m^4)^2-(m^2-m^3)^2],$

$R_{12}=\tilde{\sigma}_{24}(3b_4+m^{23})-\tilde{\sigma}_{23}(3b_3-m^{24})+\tilde{\sigma}_{34}(m^3+m^4)-b_2(2\tilde{\theta}_2-\tilde{\theta}_3+\tilde{\theta}_4)+m^{34}(\tilde{\theta}_3+\tilde{\theta}_4),$

$R_{13}=\tilde{\sigma}_{34}(3b_4-m^{23})-\tilde{\sigma}_{23}(3b_2+m^{34})-\tilde{\sigma}_{24}(m^2+m^4)+b_3(\tilde{\theta}_2-2\tilde{\theta}_3-\tilde{\theta}_4)-m^{24}(\tilde{\theta}_2+\tilde{\theta}_4),$

$R_{14}=\tilde{\sigma}_{24}(m^{34}-3b_2)-\tilde{\sigma}_{34}(m^{24}+3b_3)-\tilde{\sigma}_{23}(m^2-m^3)+b_4(\tilde{\theta}_2+\tilde{\theta}_3+2\tilde{\theta}_4)+m^{23}(\tilde{\theta}_2-\tilde{\theta}_3).$

$R_{23}=-\tilde{\sigma}_{23,1}-\tilde{\sigma}_{23}\tilde{\theta}-\tilde{\Omega}^2\tilde{\sigma}_{24}+\tilde{\Omega}^3\tilde{\sigma}_{34}-\tilde{\Omega}^4(\tilde{\theta}_2-\tilde{\theta}_3)$
$-b_2m^{24}+b_3m^{34}-b_4(m^2-m^3)+2m^{24}m^{34}-m^{23}(m^2+m^3+m^4),$

$R_{24}=-\tilde{\sigma}_{24,1}-\tilde{\sigma}_{24}\tilde{\theta}-\tilde{\Omega}^2\tilde{\sigma}_{23}+\tilde{\Omega}^3(\tilde{\theta}_2+\tilde{\theta}_4)+\tilde{\Omega}^4\tilde{\sigma}_{34}$
$+b_2m^{23}-b_3(m^2+m^4)-b_4m^{34}+2m^{23}m^{34}+m^{24}(m^2-m^3-m^4),$

$R_{34}=-\tilde{\sigma}_{34,1}-\tilde{\sigma}_{34}\tilde{\theta}-\tilde{\Omega}^2(\tilde{\theta}_3+\tilde{\theta}_4)+\tilde{\Omega}^3\tilde{\sigma}_{23}-\tilde{\Omega}^4\tilde{\sigma}_{24}$
$+b_2(m^3+m^4)-b_3m^{23}+b_4m^{24}+2m^{23}m^{24}-m^{34}(m^2-m^3+m^4).$

Appendix 2

Jacobi identities for real null tetrad basis

$J\{1,A\}$ $m^{AB}b_B=0,\qquad A,B=2,3,4,$

$J\{2,2\}$ $m^{2}{}_{,1}-m^{2}(2\hat{\theta}_{34}+\hat{\theta}_2)-2m^{23}(\hat{\theta}_{23}+\hat{\Omega}^4)-2m^{24}(\hat{\theta}_{24}-\hat{\Omega}^3)=0,$

$J\{3,3\}$ $m^{3}{}_{,1}+m^{3}(\hat{\theta}_2-2\hat{\Omega}^2)+2m^{23}(\hat{\theta}_{24}+\hat{\Omega}^3)+2m^{34}\hat{\theta}_4=0,$

$J\{4,4\}$ $\quad m^4{}_{,1}+m^4(\hat\theta_2+2\hat\Omega^2)+2m^{24}(\hat\theta_{23}-\hat\Omega^4)+2m^{34}\hat\theta_3=0,$

$J(2,3)$ $\quad m^{23}{}_{,1}-m^{23}(\hat\theta_{34}+\hat\Omega^2)+m^2(\hat\theta_{24}+\hat\Omega^3)-m^3(\hat\theta_{23}+\hat\Omega^4)$
$\qquad -m^{34}(\hat\theta_{24}-\hat\Omega^3)+m^{24}\hat\theta_4=0,$

$J(2,4)$ $\quad m^{24}{}_{,1}-m^{24}(\hat\theta_{34}-\hat\Omega^2)+m^2(\hat\theta_{23}-\hat\Omega^4)-m^4(\hat\theta_{24}-\hat\Omega^3)$
$\qquad -m^{34}(\hat\theta_{23}+\hat\Omega^4)+m^{23}\hat\theta_3=0,$

$J(3,4)$ $\quad m^{34}{}_{,1}+m^{34}\hat\theta_2+m^3\hat\theta_3+m^4\hat\theta_4+m^{23}(\hat\theta_{23}-\hat\Omega^4)+m^{24}(\hat\theta_{24}+\hat\Omega^3)=0,$

$J[3,2]$ $\quad b_{4,1}-b_4(\hat\theta_{34}+\hat\Omega^2)-b_3\hat\theta_4+b_2(\hat\theta_{24}-\hat\Omega^3)=0,$

$J[2,4]$ $\quad b_{3,1}-b_3(\hat\theta_{34}-\hat\Omega^2)-b_4\hat\theta_3+b_2(\hat\theta_{23}+\hat\Omega^4)=0,$

$J[4,3]$ $\quad b_{2,1}+b_2\hat\theta_2-b_3(\hat\theta_{24}+\hat\Omega^3)-b_4(\hat\theta_{23}-\hat\Omega^4)=0.$

Ricci tensor (wrt real null basis)

$$R_{11}=-\hat\theta_{,1}-2\hat\theta_3\hat\theta_4-2(\hat\theta_{34})^2-(\hat\theta_2)^2+4\hat\theta_{23}\hat\theta_{24},$$

$$R_{22}=-\hat\theta_{2,1}-\hat\theta_2\theta+2\hat\theta_{23}\hat\Omega^3-2\hat\theta_{24}\hat\Omega^4+2m^3m^4-\tfrac{1}{2}(m^2)^2$$
$$\qquad +2m^{23}b_3-2m^{24}b_4+4b_3b_4-2(b_2)^2,$$

$$R_{33}=-\hat\theta_{3,1}-\hat\theta_3(\hat\theta+2\hat\Omega^2)-2\hat\theta_{23}\hat\Omega^4+m^4(2b_2+2m^{34}+m^2)+2m^{24}(b_3-m^{24}),$$

$$R_{44}=-\hat\theta_{4,1}-\hat\theta_4(\hat\theta-2\hat\Omega^2)+2\hat\theta_{24}\hat\Omega^3+m^3(2m^{34}+m^2-2b_2)-2m^{23}(b_4+m^{23}),$$

$$R_{12}=\hat\theta_{23}(3b_4+m^{23})+\hat\theta_{24}(3b_3-m^{24})+m^3\hat\theta_3-m^4\hat\theta_4-2b_2(\hat\theta_2+\hat\theta_{34}),$$

$$R_{13}=-\hat\theta_{24}m^4-\hat\theta_{23}(3b_2+m^2+m^{34})+\hat\theta_3(3b_4-m^{23})+(\hat\theta_2+\hat\theta_{34})(b_3-m^{24}),$$

$$R_{14}=\hat\theta_{23}m^3+\hat\theta_{24}(m^2+m^{34}-3b_2)+\hat\theta_4(3b_3+m^{24})+(\hat\theta_2+\hat\theta_{34})(b_4+m^{23}),$$

$$R_{23}=-\hat\theta_{23,1}-\hat\theta_{23}(\hat\theta+\hat\Omega^2)-\Omega^4(\hat\theta_2+\hat\theta_{34})+\hat\theta_3\hat\Omega^3+m^2(m^{24}-b_3)+2m^4(b_4-m^{23}),$$

$$R_{24}=-\hat\theta_{24,1}-\hat\theta_{24}(\hat\theta-\hat\Omega^2)+\hat\Omega^3(\hat\theta_2+\hat\theta_{34})-\hat\theta_4\hat\Omega^4+m^2(b_4+m^{23})-2m^3(b_3+m^{24}),$$

$$R_{34}=-\hat\theta_{34,1}-\hat\theta_{34}\hat\theta+\hat\theta_{23}\hat\Omega^3-\hat\theta_{24}\Omega^4+2m^{23}m^{24}-m^2m^{34}$$
$$\qquad -\tfrac{1}{2}(m^2)^2+b_3m^{23}-b_4m^{24}-4b_3b_4+2(b_2)^2.$$

References

Barnes A 1978 *J. Phys. A: Math. Gen.* **11** 1303–14
Cohn P M 1957 *Lie Groups* (Cambridge: University Press)
Estabrook F B, Wahlquist H D and Behr C G 1968 *J. Math. Phys.* **9** 497–504
Ellis G F R 1967 *J. Math. Phys.* **8** 1171–94
Ellis G F R and MacCallum M A H 1969 *Commun. Math. Phys.* **12** 108–41
—— 1973 in *Cargèse Lectures in Physics* vol. 6 (New York: Gordon and Breach)
Greenberg P J 1970 *J. Math. Anal. Applic. (USA)* **30** 128–43
Kramer D, Stephani H, MacCallum M A H and Herlt E 1980 *Exact Solutions of Einstein's Field Equations* (Berlin: VEB Deutscher Verlag der Wissenschaften)
MaCallum M A H 1972 *Phys. Lett.* **A40** 385–6
—— 1979 in *Proc. First Int. Cracow School of Cosmology*, ed. M Demianski (Berlin: Springer)
—— 1980 in *Essays in General Relativity*, ed. F J Tipler (New York: Academic)
Ozsváth I 1965 *Akad. Wiss. Lit., Math.-Nat. Kl.* no. 1 (Mainz)
Ryan M P and Shepley L C 1975 *Homogeneous Relativistic Cosmologies* (Princeton: University Press)
Stewart J M and Ellis G F R 1968 *J. Math. Phys.* **9** 1072–82